Chapter 5

Continuity and Differentiability

Continuity

Definition

Continuity at a Point: A function f is **continuous at** c if the following three conditions are met.

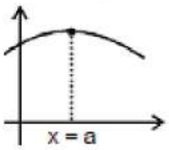
- f(x) is defined.
- $\lim_{x \to c} f(x)$ *exists*
- $\lim_{x \to c} f(x) = f(c)$

In other words function f(x) is said to be continuous at x = c , if

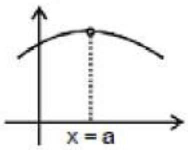
$$\lim_{x \to c} f(x) = f(c)$$

Symbolically f is continuous at x = c

if $\underset{h \to 0}{\text{Limit}} f(c - h) = \underset{h \to 0}{\text{Limit}} f(c+h) = f(c)$.



(continuous)



(Discontinuous)

One-sided Continuity

- A function f defined in some neighbourhood of a point c for c ⇒ c is said to be continuous at c from the left if
  $$\lim_{x \to c^-} f(x) = f(c)$$
- A function f defined in some neighbourhood of a point c for x ^{3} c is said to be continuous at c from the right if
  $$\lim_{x \to c^-} f(x) = f(c)$$

- One-sided continuity is a collective term for functions continuous from the left or from the right.
- If the function f is continuous at c, then it is continuous at c from the left and from the right . Conversely, if the function f is continuous at c from the left and from the right, then
  $\lim_{x \to c} f(x)$ exists & $\lim_{x \to c} f(x) = f(c)$
- The last equality means that f is continuous at c.
- If one of the one-sided limits does not exist, then $\lim_{x \to c} f(x)$ does not exist either. In this case, the point c is a discontinuity in the function, since the continuity condition is not met.

Continuity In An Interval

- A function f is said to be continuous in an open interval (a , b) if f is continuous at each & every point ∈(a, b).
- A function f is said to be continuous in a closed interval [a,b] if:
  (i) f is continuous in the open interval (a , b) &
  (ii) f is right continuous at `**a**' i.e.
  $\lim_{x \to a^+} f(x) = f(a) = \textit{a finite quantity}.$
  (iii) f is left continuous at `**b**' i.e.
  $\lim_{x \to b^-} f(x) \neq f(b) = \textit{a finite quantity}.$

A function f can be discontinuous due to any of the following three reasons:

- $\lim_{x \to c} f(x)$ does not exist i.e.
  $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$
- f(x) is not defined at x= c
- $\lim_{x \to c} f(x) \neq f(c)$
- Geometrically, the graph of the function will exhibit a break at x= c.

Example 1. Test the following functions for continuity

(a) $2x^5 - 8x^2 + 11 / x^4 + 4x^3 + 8x^2 + 8x + 4$
(b) $f(x) = 3\sin^3 x + \cos^2 x + 1 / 4\cos x - 2$

Solution.

(a) A function representing a ratio of two continuous functions will be (polynomials in this case) discontinuous only at points for which the denominator zero. But in this case $(x^4 + 4x^3 + 8x^2 + 8x + 4) = (x^2 + 2x + 2)^2 = [(x + 1)^2 + 1]^2 > 0$ (always greater than zero)
Hence f(x) is continuous throughout the entire real line.

(b) The function f(x) suffers discontinuities only at points for which the denominator is equal to zero i.e. 4 cos x - 2 = 0 or cos x = 1/2 $\Rightarrow x = x_n = \pm \pi/3 + 2n\pi (n=0, \pm1, \pm2...)$ Thus the function f(x) is continuous everywhere, except at the point x_n.

Example 2.

$$let\ f(x) = \begin{cases} -2\sin x & if\ \ x \leqslant -\pi/2 \\ A\sin x + B & if\ \ -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \cos x & if\ \ x \geqslant \frac{\pi}{2} \end{cases}$$

Find A and B so as to make the function continuous.

Solution. At x = $-\pi/2$

$\lim_{x\to -\frac{\pi}{2}^-} (-2\sin x)$ R.H.L. = $\lim_{x\to -\frac{\pi}{2}^+} A\sin x + B$

$-\pi/2 - h$
where $h \to 0$
Replace x by $-\pi/2+h$
where $h \to 0$

$$\lim_{h\to 0} -2\sin\left(-\frac{\pi}{2}-h\right) = 2 = \lim_{h\to 0} A\sin\left(-\frac{\pi}{2}+h\right) + B = B - A$$

So B - A = 2 ...(i)
At $x = \pi/2$

$\lim_{x\to \frac{\pi}{2}^-} A\sin x + B$ R.H.L. = $\lim_{x\to \frac{\pi}{2}^-} \cos x$

Replace x by $\pi/2 - h$
Replace x by $\pi/2+h$

where $h \to 0$

$$\lim_{h\to 0} A\sin\left(\frac{\pi}{2}-h\right)+B = A + B = \lim_{h\to 0} \cos\left(\frac{\pi}{2}+h\right) = 0$$

So $A+B=0$ ...(ii)

Solving (i) & (ii), $B=1, A=-1$

Example 3. Test the continuity of f(x) at x = 0 if

$$f(x) = \begin{cases} (x+1)^{2-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

Solution. For $x < 0$,

$$\text{L.H.L.} = \lim_{x\to 0-} f(x) = \lim_{h\to 0} f(0-h)$$

$$= \lim_{h\to 0} (0-h+1)^{2-\left(\frac{1}{|0-h|}+\frac{1}{(0-h)}\right)} = \lim_{h\to 0} (1-h)^2 = (1-0)^2 = 1$$

$$f(0) = 0. \ \& \ \text{R.H.L} = \lim_{x\to 0+} f(x) = \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0} (h+1)^{2-\left(\frac{1}{|h|}+\frac{1}{h}\right)} = \lim_{h\to 0} (h+1)^{2-\frac{2}{h}} = 1^{-\infty} = 1$$

L.H.L. = R.H.L. $\neq$ f(0) Hence f(x) is discontinuous at x = 0.

Example 4. If f(x) be continuous function for all real values of x and satisfies; $x^2 + \{f(x) - 2\}x + 2\sqrt{3} - 3 - \sqrt{3}\,.\,f(x) = 0$, for $x \in R$. Then find the value of $f(\sqrt{3})$.

Solution. As f(x) is continuous for all $x \in R$.

Thus,

$$\lim_{x\to\sqrt{3}} f(x) = f(3)$$

where

$f(x) = x^2 - 2x + 2\sqrt{3} - 3 / \sqrt{3} - x, x \neq \sqrt{3}$

$$\lim_{x\to\sqrt{3}} f(x) = \lim_{x\to\sqrt{3}} \frac{x^2-2x+2\sqrt{3}-3}{\sqrt{3}-x}$$

$$= \lim_{x \to \sqrt{3}} \frac{(2-\sqrt{3}-x)(\sqrt{3}-x)}{(\sqrt{3}-x)} = 2(1-\sqrt{3})$$

$f(\sqrt{3}) = 2(1-\sqrt{3})$.

Example 5.

$$Let\ f(x) = \begin{cases} \frac{1+a\cos 2x+b\cos 4x}{x^2 \sin^2 x} & if \quad x \neq 0 \\ c & if \quad x = 0 \end{cases}$$

If $f(x)$ is continuous at $x = 0$, then find the value of $(b+c)^3$-3a.

Solution.

$$\lim_{x \to 0} \frac{1+a\cos 2x+b\cos 4x}{x^4} \quad \text{as } x \to 0,$$

$N^r \to 1 + a + b$ $D^r \to 0$
for existence of limit $a + b + 1 = 0$

$$\therefore c = \lim_{x \to 0} \frac{a\cos 2x + b\cos 4x - (a+b)}{x^4} \qquad .....(2)$$

$$= -\lim_{x \to 0} \frac{\frac{a(1-\cos 2x)}{x^2} + \frac{b(1-\cos 4x)}{x^2}}{x^2}$$

limit of $N^r \Rightarrow 2a+8b = 0 \Rightarrow a = -4b$
hence
$-4b+b = -1$
$\Rightarrow b = 1/3$ and $a = -4/3$

$$\text{hence } c = \lim_{x \to 0} \frac{4(1-\cos 2x)-(1-\cos 4x)}{3x^2}$$

$= 8 \sin^2 x - 2 \sin^2 2x / 3x^4 = 8 \sin^2 x - 8\sin^2 x\cos^2 x / 3x^4$

$= 8 / 3 . \sin^2 x / x^2 . \sin^2 x / x^2 = 8 / 3$

$\Rightarrow e^A = 1 / 2 (e^{2x} A / x + B / x) \Rightarrow x . e^A = 1 / 2 (e^{2x} . A + B)$

Example 6.

$$Let\ \ f(x) = \begin{cases} \frac{a(1-x\sin x)+b\cos x+5}{x^2} & x<0 \\ 3 & x=0 \\ \left(1+\left(\frac{cx+dx^3}{x^2}\right)\right)^{\frac{1}{x}} & x>0 \end{cases}$$

If f is continuous at x = 0, then find the values of a, b, c & d.
Solution.

$$f(0^-) = \underset{x\to 0}{\text{Limit}} \ \frac{a(1-x\ \sin x)+b\cos x+5}{x^2},$$

for existence of limit $a + b + 5 = 0$

$$= \underset{x\to 0}{\text{Limit}} \ \frac{a(1-x\ \sin x)-(a+5)\cos x+5}{x^2}$$

$$= \underset{x\to 0}{\text{Limit}} \ \frac{a(1-\cos x)+5(1-\cos x)-ax\sin x}{x^2}$$

$= a / 2 + 5 / 2 - a = 3$
$\Rightarrow a = -1 \Rightarrow b = -4$

$$f(0^+) = \underset{x\to 0}{\text{Limit}} \left[1+\frac{x(c+dx^2)}{x^2}\right]^{1/x}$$

for existence of limit $c = 0$

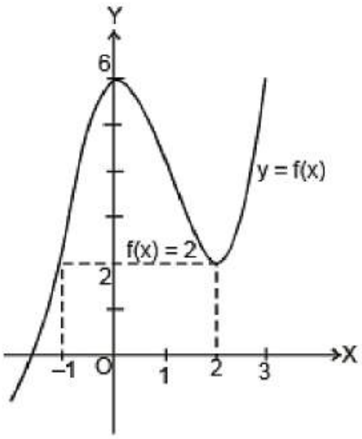
$$\underset{x\to 0}{\text{Limit}} \ (1+dx)^{1/x} = e^{\underset{x\to 0}{\text{Limit}} \frac{1}{x} dx}$$

$= ed = 3 \Rightarrow d = \ln 3$

Example 7. Let $f(x) = x^3 = 3x^2 + 6\ \forall\ x \in R$ and

$$\mathbf{g(x)} = \begin{cases} max\ \{f(t) : x+1 \le t \le 2, -3 \le x \le 0\} \\ \quad 1-x \qquad\qquad for \qquad\qquad x \ge 0 \end{cases}$$

Test continuity of g (x) for $x \in [-3, 1]$.
Solution. Since $f(x) = x^3 - 3x^2 + 6 \Rightarrow f'(x) = 3x^2 - 6x = 3x(x - 2)$ for maxima and minima $f'(x) = 0$



x = 0, 2
$f''(x) = 6x - 6$
$f''(0) = -6 < 0$ (local maxima at x = 0)
$f''(2) = 6 > 0$ (local minima at x = 2)
$x^3 - 3x^2 + 6 = 0$ has maximum 2 positive and 1 negative real roots. f(0) = 6.
Now graph of f(x) is :
Clearly f(x) is increasing in $(-\propto, 0) \cup (2, \propto)$ and decreasing in (0, 2)
$\Rightarrow x + 2 < 0 \Rightarrow x < -2 \Rightarrow -3 \le x < -2$
$\Rightarrow -2 \le x + 1 < -1$ and $-1 \le x + 2 < 0$
in both cases f(x) increases (maximum) of g(x) = f(x + 2)
$g(x) = f(x + 2); -3 \le x < -2$ ...(1)
and if $x + 1 < 0$ and $0 \le x + 2 < 2$
$-2 \le x < -1$ then g(x) = f(0)

Now for $x + 1 \ge 0$ and $x + 2 < 2 \Rightarrow -1 \le x < 0$, g(x) = f(x + 1)

$$\text{Hence } g(x) = \begin{cases} f(x+2) & ; \quad -3 \le x < -2 \\ f(0) & ; \quad -2 \le x < -1 \\ f(x+1) & ; \quad -1 \le x < +0 \\ 1 - x & ; \quad x \ge 0 \end{cases}$$

Hence g(x) is continuous in the interval [–3, 1].

Example 8. Given the function,

$f(x) = x\,[\,1 / x(1 + x) + 1 / (1 + x)(1 + 2x) + 1 / (1 + 2x)(1 + 3x) + \ldots$upto ∞

Find f (0) if f (x) is continuous at x = 0.

Solution.

$$f(x)=\frac{1}{1+x}+\frac{(1+2x)-(1+x)}{(1+x)(1+2x)}+\frac{(1+3x)-(1+2x)}{(1+2x)}+....+\frac{(1+nx)-(1+\overline{n-1}x)}{(1+\overline{n-1}x)(1+nx)}$$

f(x) = 2 / 1 + x - 1 / 1 + nx upto n terms when $x \neq 0$.

Hence

$$f(x)=\begin{cases}\frac{2}{1+x} & \text{if } x\neq 0 \text{ and } n\to\infty \\ 2 & \text{if } x=0 \text{ for continuity.}\end{cases}$$

Example 9. Let f: R →R be a function which satisfies $f(x+y^3) = f(x) + (f(y))^3$ ∀ x, y ∈ R. If f is continuous at x = 0, prove that f is continuous every where.

Solution.

To prove

$$\lim_{h\to 0} f(x+h)=f(x)$$

Put x = y = 0 in the given relation $f(0) = f(0) + (f(0))^3 \Rightarrow f(0) = 0$

Since f is continuous at x = 0

To prove

$$\lim_{h\to 0} f(x+h)=f(x)$$

$$\lim_{h\to 0} f(h)=f(0)=0.$$

Now, $\lim_{h\to 0} f(x+h) = \lim_{h\to 0} f(x) + (f(h))^3$

= f(x) + 0 = f(x).

Hence f is continuous for all x ∈ R.

Theorems of Continuity

Theorem 1. If f & g are two functions that are continuous at x= c then the functions defined by $F_1(x) = f(x) \pm g(x)$; $F_2(x) = K f(x)$ K any real number ; $F_3(x) = f(x).g(x)$ are also continuous at x= c.

Further, if g (c) is not zero, then $F_4(x)=\frac{f(x)}{g(x)}$ is also continuous at x = c.

Theorem 2. If f(x) is continuous & g(x) is discontinuous at x = a then the product function ø(x) = f(x) . g(x) is not necessarily discontinuous at x = a .

e.g. $f(x) = x$ & $g(x)=\begin{cases}\sin\frac{\pi}{x} & x\neq 0 \\ 0 & x=0\end{cases}$

Theorem 3. If f(x) and g(x) both are discontinuous at $x = a$ then the product function $\phi(x) = f(x) . g(x)$ is not necessarily discontinuous at $x = a$.

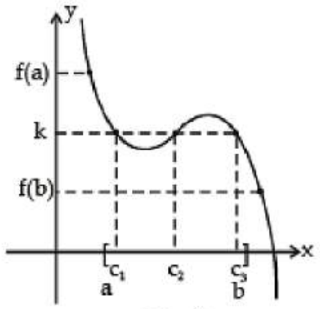
e.g. $f(x) = -g(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$

Theorem 4: Intermediate Value Theorem

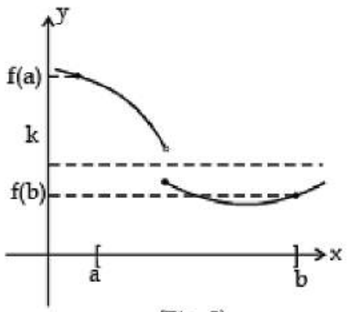
- If f is continuous on the closed interval [a, b] and k is any number between f(a) and f(b), then there is at least one number c in [a, b] such that f(c) = k.

Note:

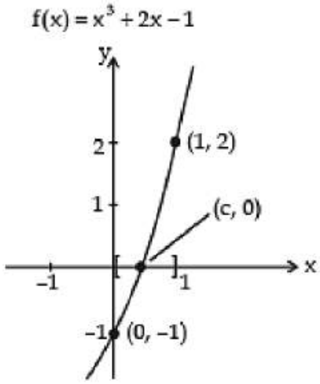
- The Intermediate Value Theorem tells that at least one c exists, but it does not give a method for finding c. Such theorems are called **existence theorems.**
- As a simple example of this theorem, consider a person's height. Suppose that a girl is 5 feet tall on her thirteenth birthday and 5 feet 7 inches tall on her fourteenth birthday. Then, for any height h between 5 feet and 7 inches, there must have been a time t when her height was exactly h. This seems reasonable because human growth is continuous and a person's height does not abruptly change from one value to another.
- The Intermediate Value Theorem guarantees the existence of at least one number c in the closed interval [a, b]. There may, of course, be more than one number c such that f(c) = k, as shown in Figure 1. A function that is not continuous does not necessarily possess the intermediate value property. For example, the graph of the function shown in Figure 2 jumps over the horizontal line given by y = k and for this function there is no value of c in [a, b] such that f(c) = k.
- The Intermediate Value Theorem often can be used to locate the zeroes of a function that is continuous on a closed interval. Specifically, if f is continuous on [a, b] and f(a) and f(b) differ in sign, then the intermediate Value Theorem guarantees the existence of at least one zero of f in the closed interval [a, b].



(Fig. 1)
f is continuous on [a, b]. (For k, there exist 3 c's.)



(Fig. 2)
f is not continuous on [a, b]. (For k, there are no c's.)



(Fig. 3)
f is continuous on [0, 1] with f(0) < 0 and f(1) > 0.

Example 10. Use the Intermediate Value Theorem to show that the polynomial function $f(x) = x^3 + 2x - 1$ has a zero in the interval [0, 1]

Sol. Note that f is continuous on the closed interval [0, 1]. Because

$f(0) = 0^3 + 2(0) - 1 = -1$ and $f(1) = 1^3 + 2(1) - 1 = 2$

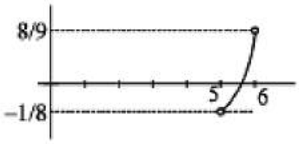
it follows that $f(0) < 0$ and $f(1) > 0$. You can therefore apply the Intermediate Value Theorem to conclude that there must be some c in [0, 1] such that $f(c) = 0$, as shown in Figure 3.

Example 11. State intermediate value theorem and use it to prove that the equation $\sqrt{x-5} = \frac{1}{x+3}$ has at least one real root.

Sol. Let $f(x) = \sqrt{x-5} = \frac{1}{x+3}$ first, $f(x)$ is continuous on [5, 6]

Also $f(5) = 0 - \frac{1}{5+3} = -\frac{1}{8} < 0$, f (6)

$f(6) = 1 - \frac{1}{9} = \frac{8}{9} > 0$



Hence by intermediate value theorem É at least one value of c ∈ (5, 6) for which f (c) = 0

$\therefore \sqrt{c-5} - \frac{1}{c+3} = 0$

c is root of the equation $\sqrt{x-5} = \frac{1}{x+3}$ and $c \in (5, 6)$

Example 12. If f(x) be a continuous function in $[0, 2\pi]$ and $f(0) = f(2\pi)$ **then prove that there exists point** $c \in (0, \pi)$ such that $f(c) = f(c + \pi)$.

Sol.

Let $g(x) = f(x) - f(x + \pi)$ ....(i)

at $x = \pi$; $g(\pi) = f(\pi) - f(2\pi)$ ....(ii)

at $x = 0$, $g(0) = f(0) - f(\pi)$ ...(iii)

adding (ii) and (iii), $g(0) + g(\pi) = f(0) - f(2\pi)$

$\Rightarrow g(0) + g(\pi) = 0$ [Given $f(0) = f(2\pi) \Rightarrow g(0) = -g(\pi)$

$\Rightarrow$ g(0) and g(π) are opposite in sign.

$\Rightarrow$ There exists a point c between 0 and p such g(c) = 0 as shown in graph;

From (i) putting x = c $g(c) = f(c) - f(c + \pi) = 0$ Hence, $f(c) = f(c + \pi)$

Differentiability of a Function and Rate of Change

D. Differentiability

Definition of Tangent : If f is defined on an open interval containing c, and if the limit

$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c+\Delta x)-f(c)}{\Delta x}$ = m exists, then the line passing through (c, f(c)) with slope m is the tangent line to the graph of f at the point (c, f(c)).

The slope of the tangent line to the graph of f at the point (c, f(c)) is also called the slope of the graph of f at x = c.

The above definition of a tangent line to a curve does not cover the possibility of a vertical tangent line. For vertical tangent lines, you can use the following definition. If f is continuous at c and

$$\lim_{\Delta x \to 0} \left| \frac{f(c+\Delta x)-f(c)}{\Delta x} \right| = \infty$$

then the vertical line, x = c, passing through (c, f(c)) is a vertical tangent line to the graph of f. For example, the function shown in Figure has a vertical tangent line at (c, f(c)). If the domain of f is the closed interval [a, b], then you can extend the definition of a vertical tangent line to include the endpoints by considering continuity and limits from the right (for x = a) and from the left (for x = b).

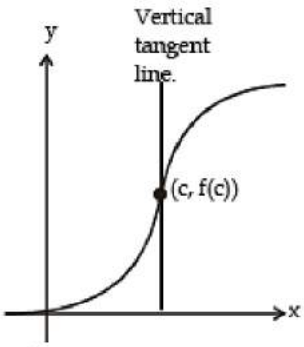


Figure
The graph of f has a vertical tanent line at (c, f(c)).

$$\lim_{\Delta x \to 0^+} \left| \frac{f(a+\Delta x)-f(a)}{\Delta x} \right| = \infty$$

$$\lim_{\Delta x \to 0^-} \left| \frac{f(b+\Delta x)-f(b)}{\Delta x} \right| = \infty$$

In the preceding section we considered the derivative of a function f at a fixed number a :

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \quad .....(1)$$

Note that alternatively, we can define

$f¢(a) = \underset{x\to a}{\text{Limit}}\frac{f(x)-f(a)}{x-a}$, provided the limit exists.

Here we change our point of view and let the number a vary. If we replace a in Equation 1 by a variable x,

we obtain $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ ...(2)

Given any number x for which this limit exists, we assign to x the number f'(x). So we can regard f' as a new function, called the **derivative of** f and defined by Equation 2. We know that the value of f'(x), can be interpreted geometrically as the slope of the tangent line to the graph of f at the point (x, f(x)).

The function f' is called the derivative of f because it has been "derived" from f by the limiting operation in Equation 2. The domain of f' is the set {x|f'(x) exists} and may be smaller than the domain of f.

Average And Instantaneous Rate Of Change

Suppose y is a function of x, say y = f(x). Corresponding to a change from x to x + Δx, the variable y changes from f(x) to f(x + Δx). The change in y is Δy = f(x + Δx) – f(x), and the average rate of change of y with respect to x is

$$\text{Average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

As the interval over which we are averaging becomes shorter (that is, as $\Delta x \to 0$), the average rate of change approaches what we would intuitively call the **instantaneous rate of change of y with respect to x,** and the difference quotient approaches the derivative $\frac{dy}{dx}$. Thus, we have

Instantaneous Rate of Change = $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x)$

To summarize :

Instantaneous Rate of Change

Suppopse f(x) is differentiable at $x = x_0$. Then the **instantaneous rate of cange** of $y = f(x)$ with respect to x at x_0 is the value of the derivative of f at x_0. That is

Instantaneous Rate of Change = $f'(x_0) = \left.\frac{dy}{dx}\right|_{x=x_0}$

Ex.13 Find the rate at which the function $y = x^2 \sin x$ is changing with respect to x when $x = \pi$.

For any x, the instantaneous rate of change in the derivative,

Sol.

$$\frac{dy}{dx} = 2x \sin x + x^2 \cos x$$

Thus, the rate when $x = \pi$ is $\left.\frac{dy}{dx}\right|_{x=\pi} = 2\pi \sin \pi + \pi^2 \cos \pi = 2\pi(0) + \pi^2(-1) = -\pi^2$

The negative sign indicates that when $x = \pi$, the function is decreasing at the rate of $\pi^2 \approx 9.9$ units of y for each one-unit increase in x.

Let us consider an example comparing the average rate of change and the instantaneous rate of change.

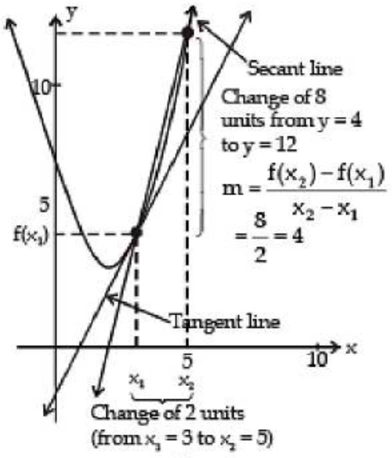
Ex.14 Let $f(x) = x^2 - 4x + 7$.

(a) Find the instantaneous rate of change of f at $x = 3$.

(b) Find the average rate of change of f with respect to

x between $x = 3$ and 5.

Sol.



Figure

(a) The derivative of the function is $f'(x) = 2x - 4$ Thus, the instantaneous rate of change of f at $x = 3$ is $f'(3) = 2(3) - 4 = 2$ The tangent line at $x = 3$ has slope 2, as shown in the figure

(b) The (average) rate of change from $x = 3$ to $x = 5$ is found by dividing the change in f by the change in x. The change in f from $x = 3$ to $x = 5$ is

$f(5) - f(3) = [5^2 - 4(5) + 7] - [3^2 - 4(3) + 7] = 8$

Thus, the average rate of change is $\frac{f(5)-f(3)}{5.3} = \frac{8}{2} = 4$

The slope of the secant line is 4, as shown in the figure.

Derivability Over An Interval

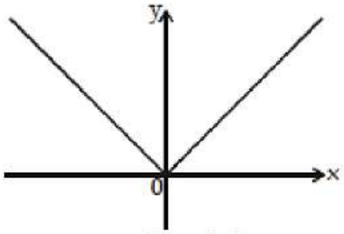
Definition : A function f is **differentiable at a** if $f'(a)$ exists. It is **differentiable on an open interval** (a,b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Derivability Over An Interval : f(x) is said to be derivable over an interval if it is derivable at each & every point of the interval. f(x) is said to be derivable over the closed interval [a, b] if :

(i) for the points a and b, $f'(a+)$ & $f'(b-)$ exist &
(ii) for any point c such that $a < c < b$, $f'(c+)$ & $f'(c-)$ exist & are equal .

How Can a Function Fail to Be Differentiable ?

We see that the function $y = |x|$ is not differentiable at 0 and Figure shows that its graph changes direction abruptly when $x = 0$. In general, if the graph of a function f has a "corner" or "kink" in it, then the graph of f has no tangent at this point and f is not differentiable there. [In trying to compute $f'(a)$, we find that the left and right limits are different.]

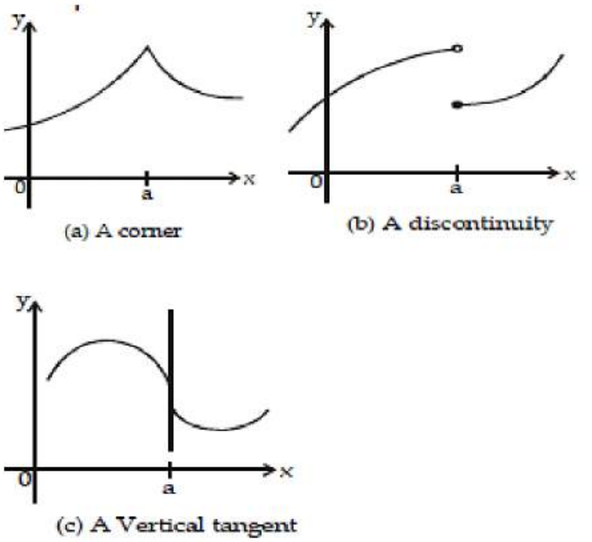


$y = f(x) = |x|$
Figure

There is another way for a function not to have a derivative. If f is discontinuous at a, then f is not differentiable at a. So at any discontinuity (for instance, a jump discontinuity), f fails to be differentiable.

A third possibility is that the curve has a **vertical tangent line** when at $x =$ a, $\lim_{x \to a} |f'(x)| = \infty$

This means that the tangent lines become steeper and steeper as $x \to a$. Figure (a, b, c) illustrates the three posibilities that we have discussed.



Right hand & Left hand Derivatives By definition : $f'(a) = \underset{h\to 0}{\text{Limit}} \dfrac{f(a+h)-f(a)}{h}$

(i) The right hand derivative of f ' at x = a denoted by $f'_+(a)$ is defined by :

$f'_+(a) = \underset{h\to 0^+}{\text{Limit}} \dfrac{f(a+h)-f(a)}{h}$, provided the limit exists & is finite.

(ii) The left hand derivative of f at x = a denoted by $f'_-(a)$ is defined by :

$f'_-(a) = \underset{h\to 0^-}{\text{Limit}} \dfrac{f(a-h)-f(a)}{-h}$, Provided the limit exists & is finite. We also write $f'_+(a) = f'(a^+)$ & $f'_-(a) = f'(a^-)$.

f'(a) exists if and only if these one-sided derivatives exist and are equal.

Ex.20 If a function f is defined by $f(x) = \begin{cases} \dfrac{xe^{1/x}}{1+e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ show that f is continuous but not derivable at x = 0

Sol. We have $f(0+0) = \lim\limits_{x\to 0+0} \dfrac{xe^{1/x}}{1+e^{1/x}} = \lim\limits_{x\to 0+0} \dfrac{e^{1/x}}{1+e^{1/x}} = \lim\limits_{x\to 0-0} \dfrac{x}{e^{1/x}+1} = 0$

$f(0-0) = \lim\limits_{x\to 0-0} \dfrac{xe^{1/x}}{1+e^{1/x}} = 0$

Also f(0) = 0 $\therefore$ f(0 + 0) = f(0 - 0) = f(0) $\Rightarrow$ f is continuous at x = 0

Again $f'(0+0) = \lim\limits_{x\to 0+0} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0+0} \dfrac{\dfrac{xe^{1/x}}{1+e^{1/x}}-0}{x} = \lim\limits_{x\to 0+0} \dfrac{e^{1/x}}{1+e^{1/x}} = \lim\limits_{x\to 0+0} \dfrac{1}{e^{-1/x}+1} = 1$

$f'(0-0) = \lim\limits_{x\to 0-0} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0-0} \dfrac{\dfrac{xe^{1/x}}{1+e^{1/x}}-0}{x} = \lim\limits_{x\to 0-0} \dfrac{e^{1/x}}{1+e^{1/x}} = 0$

Since $f'(0+0) \neq f'(0-0)$, the derivative of f(x) at x = 0 does not exist.

Ex.21 A function f(x) is such that $f\left(x+\frac{\pi}{2}\right)=\frac{\pi}{2}-|x|\ \forall\ x$. Find $f'\left(\frac{\pi}{2}\right)$**, if it exists.**

Sol. Given that = $f\left(x+\frac{\pi}{2}\right)=\frac{\pi}{2}-|x| \Rightarrow f'\left(\frac{\pi^+}{2}\right)$:

$$=\lim_{h\to 0}\frac{f\left(\frac{\pi}{2}+h\right)-f\left(\frac{\pi}{2}\right)}{h}=\frac{\frac{\pi}{2}-|h|-\frac{\pi}{2}}{h}=-1$$

$$\Rightarrow \text{and } f'\left(\frac{\pi^-}{2}\right)=\lim_{h\to 0}\frac{f\left(\frac{\pi}{2}-h\right)-f\left(\frac{\pi}{2}\right)}{-h}=\frac{\frac{\pi}{2}-|-h|-\frac{\pi}{2}}{-h}=1$$

$\Rightarrow f'\left(\frac{\pi}{2}\right)$ doesn't exist.

Ex.22 Let f **be differentiable at x = a and let** f**(a)** ^{1} **0. Evaluate** $\lim_{n\to\infty}\left\{\frac{f(a+1/n)}{f(a)}\right\}^n$.

Sol. $l = \lim_{n\to\infty}\left\{\frac{f(a+1/n)}{f(a)}\right\}^n$ (1^∞ form)

$$l = e^{\left(\lim_{n\to\infty} n\left\{\frac{f(a+1/n)-f(a)}{f(a)}\right\}\right)} = e^{\left(\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\cdot\frac{1}{f(a)}\right)} = e^{\frac{f'(a)}{f(a)}}$$ (put n = 1/h)

Ex.23 Let f : R → R satisfying $|f(x)| \le x^2,\ \forall\ x \in R$ **then show f(x) is differentiable at x = 0.**

Sol. Since, $|f(x)| \le x^2,\ \forall\ x \in R$ $\therefore$ at $x = 0$, $|f(0)| \le 0 \Rightarrow f(0) = 0$ ...(i)

$\therefore\ f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h}$ $\lim_{h\to 0}\frac{f(h)}{h}$ ....(ii) {f(0) = 0 from (i)}

Now, $\left|f\frac{(h)}{h}\right| \le |h| \Rightarrow -|h| \le f\frac{(h)}{h} \le |h| \Rightarrow \lim_{h\to 0} f\frac{(h)}{h} \to 0$ ...(iii) {using Cauchy-Squeeze theorem}

∴ from (ii) and (iii) , we get $f'(0) = 0$. i.e. $f(x)$ is differentiable at $x = 0$.

F. Operation on Differentiable Functions

1. If $f(x)$ & $g(x)$ are derivable at $x = a$ then the functions $f(x) + g(x)$, $f(x) - g(x)$, $f(x).g(x)$ will also be derivable at $x = a$ & if $g(a) \neq 0$ then the function $f(x)/g(x)$ will also be derivable at $x = a$.

If f and g are differentiable functions, then prove that their product fg is differentiable.

Let a be a number in the domain of fg. By the definition of the product of two functions we have

$(fg)(a) = f(a)g(a)$ $(fg)(a + t) = f(a + t)g(a + t)$.

Hence $(fg)'(a) = \lim_{t\to 0}\frac{f(g)(a+t)-(fg)(a)}{t} = \lim_{t\to 0}\frac{f(a+t)g(a+t)-f(a)g(a)}{t}$

The following algebraic manipulation will enable us to put the above fraction into a form in which we can see what the limit is:

$f(a + t)g(a + t) - f(a)g(a) = f(a + t)g(a + t) - f(a)g(a + t) + f(a)g(a + t) - f(a)g(a)$

$= [f(a + t) - f(a)]g(a + t) + [g(a + t) - g(a)]f(a)$.

Thus $(fg)'(a) = \lim_{t\to 0}\left[\frac{f(a+t)-f(a)}{t}g(a+t)+\frac{g(a+t)-g(a)}{t}f(a)\right]$.

The limit of a sum of products is the sum of the products of the limits. Moreover, $f'(a)$ and $g'(a)$ exist by hypothesis. Finally, since g is differentiable at a, it is continuous there ; and so $\lim_{t\to 0} g(a + t) = f(a)$. We conclude that

$(fg)'(a) = \lim_{t\to 0}\left[\frac{f(a+t)-f(a)}{t}g(a+t)+\frac{g(a+t)-g(a)}{t}f(a)\right]$.

$= f'(a)g(a) + g'(a)f(a) = (f'g + g'f)(a)$.

2. If $f(x)$ is differentiable at $x = a$ & $g(x)$ is not differentiable at $x = a$, then the product function $F(x) = f(x).g(x)$ can still be differentiable at $x = a$ e.g. $f(x) = x$ and $g(x) = |x|$.

3. If f(x) & g(x) both are not differentiable at x = a then the product function ;

F(x) = f(x) . g(x) can still be differentiable at x = a e.g. f(x) = $|x|$ & g(x) = $|x|$

4. If f(x) & g(x) both are non-deri. at x = a then the sum function F(x) = f(x) + g(x) may be a differentiable function . e.g. f(x) = $|x|$ & g(x) = - $|x|$.

5. If f(x) is derivable at x = a ⇒ f '(x) is continuous at x = a.

e.g. $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

G. Functional Equations

Ex.24 Let f(xy) = xf(y) + yf(x) for all x, $y \in R^+$ and f(x) be differentiable in (0, ∞) then determine f(x).

Given f(xy)= xf(y) + yf(x)

Sol. Replacing x by 1 and y by x then we get x f(1) = 0

$$\text{Now, } f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f\left(x\left(1+\frac{h}{x}\right)\right)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{xf\left(1+\frac{h}{x}\right)+\left(1+\frac{h}{x}\right)f(x)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{xf\left(1+\frac{h}{x}\right)+\frac{h}{x}f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)}{\left(\frac{h}{x}\right)}+\lim_{h\to 0}\frac{f(x)}{x} = f'(x) + \frac{f(x)}{x}$$

$$\Rightarrow f'(x) - \frac{f(x)}{x} = f'(1) \Rightarrow \frac{xf'(x)-f(x)}{x^2} = \frac{f'(1)}{x}$$

$$\Rightarrow \frac{d}{dx}\left\{\frac{f(x)}{x}\right\} = \frac{f'(1)}{x}$$

On integrating w.r.t.x and taking limit 1 to x then f(x)/x - f(1)/1 = f'(1) (ln x – ln 1)

$$\Rightarrow \frac{f(x)}{x} - 0 = f'(1)\ln x$$

$\therefore$ f(1) = 0) $\therefore$ f(x) = f'(1) (x ln x)

Alternative Method :

Given f(xy) = xf(y) + yf(x)

Differentiating both sides w.r.t.x treating y as constant, f'(xy) . y = f(y) + yf'(x)

Putting y = x and x = 1, then

f'(xy). x = f(x) + xf'(x)

$$\Rightarrow \frac{xf'(x) - f(x)}{x^2} = \frac{f'(1)}{x}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{f(x)}{x}\right) = \frac{f'(1)}{x}$$

Integrating both sides w.r.t.x taking limit 1 to x,

$$\frac{f(x)}{x} - \frac{f(1)}{1} = f'(1)\{\ln x - \ln 1\} \Rightarrow \frac{f(x)}{x} - 0 = f'(1)\ln x$$

Hence, f(x) =- f'(1)(x ln x).

Ex.25 If $e^{-xy}f(xy) = e^{-x}f(x) + e^{-y}f(y)\ \forall x, y \in R^+$, **and f'(1) = e, determine f(x).**

Sol.

Given e^{-xy} f(xy) = e^{-x}f(x) + e^{-y}f(y) ....(1)

Putting x = y = 1 in (1), we get f(1) = 0 ...(2)

Now, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{f\left(x\left(1+\frac{h}{x}\right)\right)-f(x.1)}{h}$$

$$= \lim_{h\to 0} \frac{e^{x+h}\left\{e^{-x}f(x)+e^{-1\frac{h}{x}}f\left(1+\frac{h}{x}\right)\right\}-2^{x}(e^{-x}f(x)+e^{-1}f(1))}{h}$$

$$= \lim_{h\to 0} \frac{e^{h}f(x)+e^{x+h-1\frac{h}{x}}f\left(1+\frac{h}{x}\right)-f(x)-e^{x-1}f(1)}{h}$$

$$= f(x)\lim_{h\to 0}\left(\frac{e^{h}-1}{h}\right) + e^{(x-1)}\lim_{h\to 0}\frac{e^{h-\frac{h}{x}}f\left(1+\frac{h}{x}\right)}{x.\frac{h}{x}} \quad (\because f(1)=0)$$

$$= f(x)\,.\,1 + e^{x-1}.\frac{f'(1)}{x} = f(x) + \frac{e^{x-1}.e}{x} \quad (\because f'(1)=e)$$

$$f'(x) = f(x) + \frac{e^{x}}{x} \quad \Rightarrow \quad e^{-x}f'(x) - e^{-x}f(x) = \frac{1}{x}$$

$$\Rightarrow \frac{d}{dx}(e^{-x}f(x)) = \frac{1}{x}$$

On integrating we have $e^{-x}f(x) = \ln x + c$ at $x = 1$, $c = 0$

$\therefore f(x) = ex \ln x$

Ex.26 Let f be a function such that $f(x + f(y)) = f(f(x)) + f(y)\, x$, $y\ \forall\ x, y \in R$ and $f(h) = h$ for $0 < h < \varepsilon$ where $\varepsilon > 0$, then determine $f''(x)$ and $f(x)$.

Sol. Given $f(x + f(y)) = f(f(x) + f(y))$ .....(1)

Put $x = y = 0$ in (1), then $f(0 + f(0)) = f(f(0)) + f(0) \Rightarrow f(f(0)) = f(f(0)) + f(0)$

$\therefore f(0) = 0$ ...(2)

Now $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \frac{f(f(h))}{h}$ {from (1)}

$= \lim_{h\to 0} \frac{f(h)}{h}$ $(\because f(h) = h)$ $= \lim_{h\to 0} \frac{h}{h} = 1.$

Integrating both sides with limites 0 to x then $f(x) = x \;\; \therefore f'(x) = 1.$

Theorems of Continuity

C. Theorems of Continuity

THEOREM–1 If f & g are two functions that are continuous at x= c then the functions defined by $F_1(x) = f(x) \pm g(x)$; $F_2(x) = K\,f(x)$ K any real number ; $F_3(x) = f(x).g(x)$ are also continuous at x= c.

Further, if g (c) is not zero, then $F_4(x) = \frac{f(x)}{g(x)}$ is also continuous at $x = c$.

THEOREM–2 If f(x) is continuous & g(x) is discontinuous at $x = a$ then the product function $\phi(x) = f(x) \,.\, g(x)$ is not necessarily discontinuous at $x = a$.

e.g. $f(x) = x$ & $g(x) = \begin{cases} \sin\frac{\pi}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

THEOREM–3 If f(x) and g(x) both are discontinuous at $x = a$ then the product function $\phi(x) = f(x) \,.\, g(x)$ is not necessarily discontinuous at $x = a$.

e.g. $f(x) = -g(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$

Theorems-4 : Intermediate Value Theorem

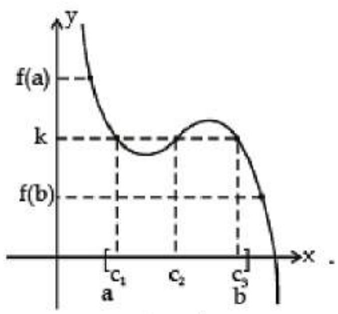
If f is continuous on the closed interval [a, b] and k is any number between f(a) and f(b), then there is at least one number c in [a, b] such that $f(c) = k$.

Note that the Intermediate Value Theorem tells that at least one c exists, but it does not give a method for finding c. Such theorems are called **existence theorems.**

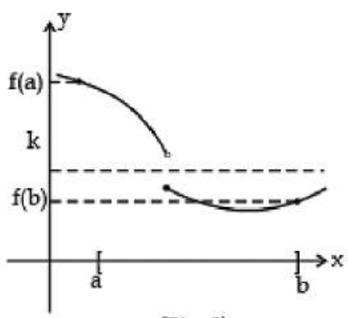
As a simple example of this theorem, consider a person's height. Suppose that a girl is 5 feet tall on her thirteenth birthday and 5 feet 7 inches tall on her fourteenth birthday. Then, for any height h between 5 feet and 7 inches, there must have been a time t when her height was exactly h. This seems reasonable because human growth is continuous and a person's height does not abruptly change from one value to another.

The Intermediate Value Theorem guarantees the existence of at least one number c in the closed interval [a, b]. There may, of course, be more than one number c such that f(c) = k, as shown in Figure 1. A function that is not continuous does not necessarily possess the intermediate value property. For example, the graph of the function shown in Figure 2 jumps over the horizontal line given by y = k and for this function there is no value of c in [a, b] such that f(c) = k.

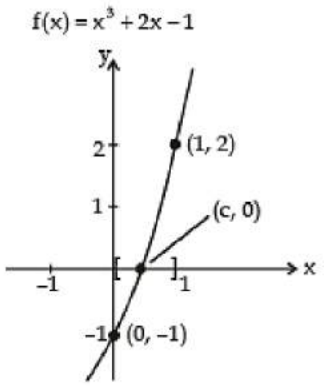
The Intermediate Value Theorem often can be used to locate the zeroes of a function that is continuous on a closed interval. Specifically, if f is continuous on [a, b] and f(a) and f(b) differ in sign, then the intermediate Value Theorem guarantees the existence of at least one zero of f in the closed interval [a, b].



(Fig. 1)
f is continuous on [a, b]. (For k, there exist 3 c's.)



(Fig. 2)
f is not continuous on [a, b]. (For k, there are no c's.)



(Fig. 3)
f is continuous on [0, 1] with $f(0) < 0$ and $f(1) > 0$.

Ex.10 Use the Intermediate Value Theorem to show that the polynomial function $f(x) = x^3 + 2x - 1$ has a zero in the interval [0, 1]

Sol. Note that f is continuous on the closed interval [0, 1]. Because

$f(0) = 0^3 + 2(0) - 1 = -1$ and $f(1) = 1^3 + 2(1) - 1 = 2$

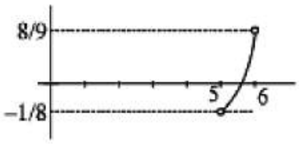
it follows that $f(0) < 0$ and $f(1) > 0$. You can therefore apply the Intermediate Value Theorem to conclude that there must be some c in [0, 1] such that $f(c) = 0$, as shown in Figure 3.

Ex.11 State intermediate value theorem and use it to prove that the equation $\sqrt{x-5} = \frac{1}{x+3}$ has at least one real root.

Sol. Let $f(x) = \sqrt{x-5} = \frac{1}{x+3}$ first, $f(x)$ is continuous on [5, 6]

Also $f(5) = 0 - \frac{1}{5+3} = -\frac{1}{8} < 0$, f (6)

$f(6) = 1 - \frac{1}{9} = \frac{8}{9} > 0$



Hence by intermediate value theorem É at least one value of c ∈ (5, 6) for which f (c) = 0

$\therefore \sqrt{c-5} - \frac{1}{c+3} = 0$

c is root of the equation $\sqrt{x-5} = \frac{1}{x+3}$ and $c \in (5, 6)$

Ex.12 If f(x) be a continuous function in $[0, 2\pi]$ and $f(0) = f(2\pi)$ **then prove that there exists point** $c \in (0, \pi)$ such that $f(c) = f(c + \pi)$.

Sol.

Let $g(x) = f(x) - f(x + \pi)$ ....(i)

at $x = \pi$; $g(\pi) = f(\pi) - f(2\pi)$ ....(ii)

at $x = 0$, $g(0) = f(0) - f(\pi)$ ...(iii)

adding (ii) and (iii), $g(0) + g(\pi) = f(0) - f(2\pi)$

$\Rightarrow g(0) + g(\pi) = 0$ [Given $f(0) = f(2\pi) \Rightarrow g(0) = -g(\pi)$

$\Rightarrow$ g(0) and g(π) are opposite in sign.

⇒ There exists a point c between 0 and p such $g(c) = 0$ as shown in graph;

From (i) putting $x = c$ $g(c) = f(c) - f(c + \pi) = 0$ Hence, $f(c) = f(c + \pi)$

Derivability Over An Interval

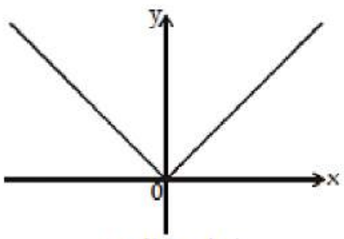
Definition : A function f is **differentiable at a** if $f'(a)$ exists. It is **differentiable on an open interval** (a,b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Derivability Over An Interval : f(x) is said to be derivable over an interval if it is derivable at each & every point of the interval. f(x) is said to be derivable over the closed interval [a, b] if :

(i) for the points a and b, $f'(a+)$ & $f'(b-)$ exist &
(ii) for any point c such that $a < c < b$, $f'(c+)$ & $f'(c-)$ exist & are equal .

How Can a Function Fail to Be Differentiable ?

We see that the function $y = |x|$ is not differentiable at 0 and Figure shows that its graph changes direction abruptly when $x = 0$. In general, if the graph of a function f has a "corner" or "kink" in it, then the graph of f has no tangent at this point and f is not differentiable there. [In trying to compute $f'(a)$, we find that the left and right limits are different.]

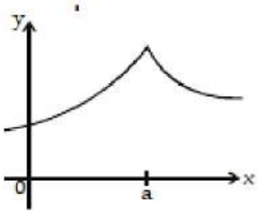


$y = f(x) = |x|$
Figure

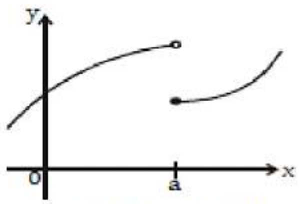
There is another way for a function not to have a derivative. If f is discontinuous at a, then f is not differentiable at a. So at any discontinuity (for instance, a jump discontinuity), f fails to be differentiable.

A third possibility is that the curve has a **vertical tangent line** when at $x = a$, $\lim_{x \to a} |f'(x)| = \infty$

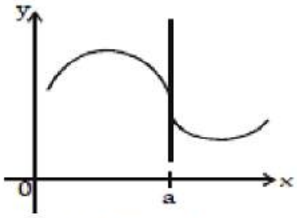
This means that the tangent lines become steeper and steeper as $x \to a$. Figure (a, b, c) illustrates the three posibilities that we have discussed.



(a) A corner



(b) A discontinuity



(c) A Vertical tangent

Right hand & Left hand Derivatives By definition : $f'(a) = \underset{h\to 0}{\text{Limit}} \dfrac{f(a+h)-f(a)}{h}$

(i) The right hand derivative of f ' at x = a denoted by $f'_+(a)$ is defined by :

$f'_+(a) = \underset{h\to 0^+}{\text{Limit}} \dfrac{f(a+h)-f(a)}{h}$, provided the limit exists & is finite.

(ii) The left hand derivative of f at x = a denoted by $f'_-(a)$ is defined by :

$f'_-(a) = \underset{h\to 0^+}{\text{Limit}} \dfrac{f(a-h)-f(a)}{-h}$, Provided the limit exists & is finite. We also write $f'_+(a) = f'(a^+)$ & $f'_-(a) = f'(a^-)$.

f'(a) exists if and only if these one-sided derivatives exist and are equal.

Ex.20 If a function f is defined by $f(x) = \begin{cases} \dfrac{xe^{1/x}}{1+e^{1/x}}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$ **show that f is continuous but not derivable at x = 0**

Sol. We have $f(0+0) = \lim\limits_{x\to 0+0} \dfrac{xe^{1/x}}{1+e^{1/x}} = \lim\limits_{x\to 0+0} \dfrac{e^{1/x}}{1+e^{1/x}} = \lim\limits_{x\to 0+0} \dfrac{x}{e^{1/x}+1} = 0$

$f(0 - 0) = \lim_{x\to 0-0} \frac{xe^{1/x}}{1+e^{1/x}} = 0$

Also $f(0) = 0$ $\therefore$ $f(0 + 0) = f(0 - 0) = f(0) \Rightarrow f$ is continuous at $x = 0$

Again $f'(0 + 0) = \lim_{x\to 0+0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0+0} \frac{\frac{xe^{1/x}}{1+e^{1/x}}-0}{x} = \lim_{x\to 0+0} \frac{e^{1/x}}{1+e^{1/x}} = \lim_{x\to 0+0} \frac{1}{e^{-1/x}+1} = 1$

$f'(0 - 0) = \lim_{x\to 0-0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0-0} \frac{\frac{xe^{1/x}}{1+e^{1/x}}-0}{x} = \lim_{x\to 0-0} \frac{e^{1/x}}{1+e^{1/x}} = 0$

Since $f'(0 + 0) \neq f'(0 - 0)$, the derivative of $f(x)$ at $x = 0$ does not exist.

Ex.21 A function f(x) is such that $f\left(x+\frac{\pi}{2}\right)=\frac{\pi}{2} - |x|\ \forall\ x.$ Find $f'\left(\frac{\pi}{2}\right)$**, if it exists.**

Sol. Given that $= f\left(x+\frac{\pi}{2}\right)=\frac{\pi}{2} - |x| \Rightarrow f'\left(\frac{\pi^+}{2}\right)$:

$= \lim_{h\to 0} \frac{f\left(\frac{\pi}{2}+h\right)-f\left(\frac{\pi}{2}\right)}{h} = \frac{\frac{\pi}{2}-|h|-\frac{\pi}{2}}{h} = -1$

$\Rightarrow$ and $f'\left(\frac{\pi^-}{2}\right) = \lim_{h\to 0} \frac{f\left(\frac{\pi}{2}-h\right)-f\left(\frac{\pi}{2}\right)}{-h} = \frac{\frac{\pi}{2}-|-h|-\frac{\pi}{2}}{-h} = 1$

$\Rightarrow f'\left(\frac{\pi}{2}\right)$ doesn't exist.

Ex.22 Let *f* be differentiable at x = a and let *f*(a) ¹ 0. Evaluate $\lim_{n\to\infty}\left\{\frac{f(a+1/n)}{f(a)}\right\}^n$.

Sol. $l = \lim_{n\to\infty}\left\{\frac{f(a+1/n)}{f(a)}\right\}^n$ $(1^\infty$ form$)$

$l = e^{\left(\lim_{n\to\infty} n\left\{\frac{f(a+1/n)-f(a)}{f(a)}\right\}\right)} = e^{\left(\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\cdot\frac{1}{f(a)}\right)} = e^{\frac{f'(a)}{f(a)}}$ (put $n = 1/h$)

Ex.23 Let f : R → R satisfying $|f(x)| \le x^2, \forall x \in R$ then show f(x) is differentiable at x = 0.

Sol. Since, $|f(x)| \le x^2, \forall x \in R$ $\therefore$ at $x = 0$, $|f(0)| \le 0 \Rightarrow f(0) = 0$ ...(i)

$\therefore$ $f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h}$ $\lim_{h\to 0}\frac{f(h)}{h}$ ....(ii) {f(0) = 0 from (i)}

Now, $\left|f\frac{(h)}{h}\right| \le |h| \Rightarrow -|h| \le f\frac{(h)}{h} \le |h| \Rightarrow \lim_{h\to 0} f\frac{(h)}{h}$ → 0 ...(iii) {using Cauchy-Squeeze theorem}

$\therefore$ from (ii) and (iii) , we get f'(0) = 0. i.e. f(x) is differentiable at x = 0.

F. Operation on Differentiable Functions

1. If f(x) & g(x) are derivable at x = a then the functions f(x) + g(x), f(x) - g(x), f(x). g(x) will also be derivable at x = a & if g (a) $\neq$ 0 then the function f(x)/g(x) will also be derivable at x = a.

If f and g are differentiable functions, then prove that their product fg is differentiable.

Let a be a number in the domain of fg. By the definition of the product of two functions we have

(fg) (a) = f(a) g(a) (fg) (a + t) = f(a + t) g(a + t).

Hence (fg)' (a) = $\lim_{t\to 0}\frac{f(g)(a+t)-(fg)(a)}{t}$ $= \lim_{t\to 0}\frac{f(a+t)g(a+t)-f(a)g(a)}{t}$

The following algebraic manipulation will enable us to put the above fraction into a form in which we can see what the limit is:

f(a + t) g(a + t) - f(a) g(a) = f(a + t) g(a + t) - f(a) g (a + t) + f(a)g(a + t) - f(a) g(a)

= [f(a + t) - f(a)] g(a + t) + [g(a + t) - g(a)] f(a).

Thus (fg)' (a) = $\lim_{t\to 0}\left[\frac{f(a+t)-f(a)}{t}g(a+t)+\frac{g(a+t)-g(a)}{t}f(a)\right]$.

The limit of a sum of products is the sum of the products of the limits. Moreover, f'(a) and g'(a) exist by hypothesis. Finally, since g is differentiable at a, it is continuous there ; and so $\lim_{t\to 0} g(a+t) = f(a)$. We conclude that

$$(fg)'(a) = \lim_{t\to 0}\left[\frac{f(a+t)-f(a)}{t}g(a+t)+\frac{g(a+t)-g(a)}{t}f(a)\right].$$

$= f'(a)g(a) + g'(a)f(a) = (f'g + g'f)(a)$.

2. If f(x) is differentiable at x = a & g(x) is not differentiable at x = a , then the product function F(x) = f(x) . g(x) can still be differentiable at x = a e.g. f(x) = x and g(x) = $|x|$.

3. If f(x) & g(x) both are not differentiable at x = a then the product function ;

F(x) = f(x) . g(x) can still be differentiable at x = a e.g. f(x) = $|x|$ & g(x) = $|x|$

4. If f(x) & g(x) both are non-deri. at x = a then the sum function F(x) = f(x) + g(x) may be a differentiable function . e.g. f(x) = $|x|$ & g(x) = - $|x|$.

5. If f(x) is derivable at x = a ⇒ f '(x) is continuous at x = a.

$$\text{e.g. } f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

G. Functional Equations

Ex.24 Let f(xy) = xf(y) + yf(x) for all x, $y \in R^+$ and f(x) be differentiable in (0, ∞) then determine f(x).

Given f(xy)= xf(y) + yf(x)

Sol. Replacing x by 1 and y by x then we get x f(1) = 0

$$\text{Now, } f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f\left(x\left(1+\frac{h}{x}\right)\right)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{xf\left(1+\frac{h}{x}\right)+\left(1+\frac{h}{x}\right)f(x)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{xf\left(1+\frac{h}{x}\right)+\frac{h}{x}f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)}{\left(\frac{h}{x}\right)}+\lim_{h\to 0}\frac{f(x)}{x} = f'(x) + \frac{f(x)}{x}$$

$$\Rightarrow f'(x) - \frac{f(x)}{x} = f'(1) \Rightarrow \frac{xf'(x)-f(x)}{x^2}=\frac{f'(1)}{x}$$

$$\Rightarrow \frac{d}{dx}\left\{\frac{f(x)}{x}\right\}=\frac{f'(1)}{x}$$

On integrating w.r.t.x and taking limit 1 to x then $f(x)/x - f(1)/1 = f'(1)\ (\ln x - \ln 1)$

$$\Rightarrow \frac{f(x)}{x} - 0 = f'(1)\ln x$$

$\therefore f(1) = 0) \therefore f(x) = f'(1)\ (x \ln x)$

Alternative Method :

Given $f(xy) = xf(y) + yf(x)$

Differentiating both sides w.r.t.x treating y as constant, $f'(xy) . y = f(y) + yf'(x)$

Putting $y = x$ and $x = 1$, then

$f'(xy). x = f(x) + xf'(x)$

$$\Rightarrow \frac{xf'(x)-f(x)}{x^2}=\frac{f'(1)}{x}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{f(x)}{x}\right)=\frac{f'(1)}{x}$$

Integrating both sides w.r.t.x taking limit 1 to x,

$$\frac{f(x)}{x}-\frac{f(1)}{1} = f'(1)\{\ln x - \ln 1\} \Rightarrow \frac{f(x)}{x} - 0 = f'(1)\ln x$$

Hence, $f(x) = -f'(1)(x \ln x)$.

Ex.25 If $e^{-xy}f(xy) = e^{-x}f(x) + e^{-y}f(y)\ \forall x, y \in R^+$, **and $f'(1) = e$, determine $f(x)$.**

Sol.

Given $e^{-xy} f(xy) = e^{-x}f(x) + e^{-y}f(y)$ ....(1)

Putting $x = y = 1$ in (1), we get $f(1) = 0$ ...(2)

$$\text{Now, } f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f\left(x\left(1+\frac{h}{x}\right)\right)-f(x.1)}{h}$$

$$= \lim_{h\to 0}\frac{e^{x+h}\left\{e^{-x}f(x)+e^{-1\frac{h}{x}}f\left(1+\frac{h}{x}\right)\right\}-2^{x}(e^{-x}f(x)+e^{-1}f(1))}{h}$$

$$= \lim_{h\to 0}\frac{e^{h}f(x)+e^{x+h-1\frac{h}{x}}f\left(1+\frac{h}{x}\right)-f(x)-e^{x-1}f(1)}{h}$$

$$= f(x)\lim_{h\to 0}\left(\frac{e^{h}-1}{h}\right) + e^{(x-1)}\lim_{h\to 0}\frac{e^{h-\frac{h}{x}}f\left(1+\frac{h}{x}\right)}{x.\frac{h}{x}} \quad (\because f(1) = 0)$$

$$= f(x)\ .\ 1 + e^{x-1}.\ \frac{f'(1)}{x} \quad = f(x) + \frac{e^{x-1}.e}{x} \quad (\because f'(1) = e)$$

$$f'(x) = f(x) + \frac{e^{x}}{x} \quad \Rightarrow \quad e^{-x}f'(x) - e^{-x} f(x) = \frac{1}{x}$$

$$\Rightarrow \frac{d}{dx}(e^{-x} f(x)) = \frac{1}{x}$$

On integrating we have $e^{-x}f(x) = \ln x + c$ at $x = 1$, $c = 0$

$\therefore\ f(x) = ex \ln x$

Ex.26 Let f be a function such that f(x + f(y)) = f(f(x)) + f(y) x, y $\forall$ x, y $\in$ R and f(h) = h for $0 < h < \varepsilon$ where $\varepsilon > 0$, then determine f''(x) and f(x).

Sol. Given $f(x + f(y)) = f(f(x) + f(y))$ .....(1)

Put x = y = 0 in (1), then $f(0 + f(0)) = f(f(0)) + f(0) \Rightarrow f(f(0)) = f(f(0)) + f(0)$

$\therefore f(0) = 0$ ...(2)

Now $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0}\frac{f(f(h))}{h}$ {from (1)}

$= \lim_{h\to 0}\frac{f(h)}{h}$ ($\because$ f(h) = h) $= \lim_{h\to 0}\frac{h}{h} = 1.$

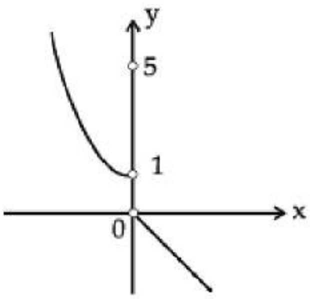
Integrating both sides with limites 0 to x then f(x) = x $\therefore f'(x) = 1$.

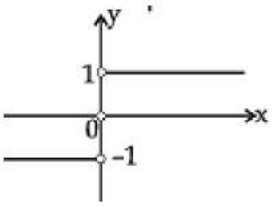
Classification of Discontinuity

Definition

- Let a function f be defined in the neighbourhood of a point c, except perhaps at c itself.
- Also, let both one–sided limits $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist, where $\lim_{x\to c^+} f(x) \neq \lim_{x\to c^-} f(x)$.
- Then the point c is called a discontinuity of the first kind in the function f(x).
- In more complicated case $\lim_{x\to c} f(x)$ may not exist because one or both one-sided limits do not exist. Such condition is called a discontinuity of the second kind.

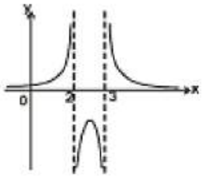
The function $y = \begin{cases} x^2+1 & \text{for} \quad x<0, \\ 5 & \text{for} \quad x=0, \\ -x & \text{for} \quad x>0, \end{cases}$





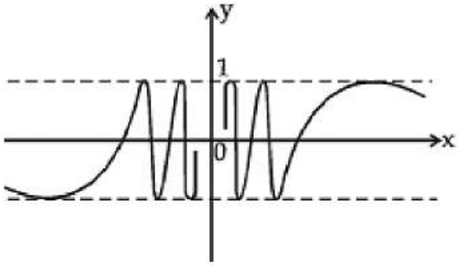
has a discontinuity of the first kind at x = 0

- The function y = |x| /x is defined for all $x \in R, x \neq 0$; but at x = 0 it has a discontinuity of the first kind.
  The left-hand limit is $\lim_{x\to 0^-} y = -1$, while the
- right-hand limit is $\lim_{x\to 0^+} y = 1$
- The function $y = \frac{1}{(x-2)(x-3)}$ has no limits (neither one-sided nor two-sided) at x = 2 and x = 3 since $\lim_{x\to 0} \frac{1}{(x-2)(x-3)} = \infty$. Therefore x = 2 and x = 3 are discontinuities of the second kind



- The function y = ln |x| at the point x = 0 has the limits $\lim_{x\to 0} \ln|x| = -\infty$. Consequently, $\lim_{x\to 0} f'(x)$ (and also the one-sided limits) do not exist; x = 0 is a discontinuity of the second kind.
- It is not true that discontinuities of the second kind only arise when $\lim_{x\to 0} \ln|x| = -\infty$.
  The situation is more complicated.

- Thus, the function $y = \sin(1/x)$, has no one-sided limits for $x \to 0^-$ and $x \to 0^+$, and does not tend to infinity as $x \to 0$ There is no limit as $x \to 0$ since the values of the function $\sin(1/x)$ do not approach a certain number, but repeat an infinite number of times within the interval from -1 to 1 as $x \to 0$.



Removable & Irremovable Discontinuity

In case $\underset{x \to c}{\text{Limit}} f(x)$ exists but is not equal to f(c) then the function is said to have a removable discontinuity. In this case we can redefine the function such that $\underset{x \to c}{\text{Limit}} f(x) = f(c)$ & make it continuous at $x = c$.

1. Removable Type of Discontinuity Can Be Further Classified as

- **Missing Point Discontinuity:** where $\underset{x \to a}{\text{Limit}} f(x)$ exists finitely but f(a) is not defined. e.g. $f(x) = \frac{(1-x)(9-x^2)}{(1-x)}$ has a missing point discontinuity at $x = 1$
- **Isolated Point Discontinuity :** where $\underset{x \to a}{\text{Limit}} f(x)$ exists & f(a) also exists but $\underset{x \to a}{\text{Limit}} \neq f(a)$.
  e.g. $f(x) = \frac{x^2 - 16}{x - 4}$, $x \neq 4$ & $f(4) = 9$ has a break at $x = 4$.

In case $\underset{x \to c}{\text{Limit}} f(x)$ does not exist then it is not possible to make the function continuous by redefining it . Such discontinuities are known as non - removable discontinuity.

2. Irremovable Type Of Discontinuity Can Be Further Classified as

- **Finite discontinuity :** e.g. $f(x) = x - [x]$ at all integral x.
- **Infinite discontinuity :** e.g. $f(x) = \frac{1}{x-4}$ or $g(x) = \frac{1}{(x-4)^2}$ at $x = 4$.
- **Oscillatory discontinuity :** e.g. $f(x) = \sin 1/x$ at $x = 0$

In all these cases the value of f(a) of the function at x= a (point of discontinuity) may or may not exist but where $\underset{x\to a}{\text{Limit}} f(x)$ does not exist.

Remark

(i) In case of finite discontinuity the non-negative difference between the value of the RHL at $x = c$ & LHL at $x = c$ is called **The Jump Of Discontinuity**. A function having a finite number of jumps in a given interval I is called a **Piece-wise Continuous** or **Sectionally Continuous** function in this interval.

(ii) All Polynomials, Trigonometrical functions, Exponential & Logarithmic functions are continuous in their domains.

(iii) Point functions are to be treated as discontinuous eg. $f(x) = \sqrt{1-x}+\sqrt{x-1}$ is not continuous at $x = 1$.

(iv) If f is continuous at $x = c$ & g is continuous at $x = f(c)$ then the composite $g[f(x)]$ is continuous at $x = c$.

eg. $f(x) = \frac{x \sin x}{x^2+2}$ & $g(x) = |x|$ are continuous at $x = 0$, hence the composite $gof\,(x) = \left|\frac{x \sin x}{x^2+2}\right|$ will also be continuous at $x = 0$.

Relation Between Continuity & Differentiability

E. Relation between Continuity & Differentiability

If a function f is derivable at x then f is continuous at x.

For : $f'(x) = \underset{h\to 0}{\text{Limit}} \frac{f(x+h)-f(x)}{h}$ exists.

Also $f(x+h)-f(x) = \frac{f(x+h)-f(x)}{h}.h \; [h \neq 0]$

Therefore $\underset{h\to 0}{\text{Limit}} [f(x+h)-f(x)]$

$= \underset{h\to 0}{\text{Limit}} \frac{f(x+h)-f(x)}{h}.h = f'(x).0 = 0$

Therefore $\lim\limits_{h \to 0} [f(x+h) - f(x)] = 0$

$\Rightarrow \lim\limits_{h \to 0} f(x+h) = f(x) \Rightarrow f$ is continuous at x.

If f(x) is derivable for every point of its domain, then it is continuous in that domain .

The converse of the above result is not true :

"If f is continuous at x, then f may or maynot be derivable at x"

The functions $f(x) = |x|$ & $g(x) = x \sin \frac{1}{x}$; $x \neq 0$ & $g(0) = 0$ are continuous at x = 0 but not derivable at x = 0.

Remark :

(a) Let $f'_{+}(a) = p$ & $f'_{-}(a) = q$ where p & q are finite then :

(i) $p = q \Rightarrow$ f is derivable at $x = a \Rightarrow$ f is continuous at $x = a$.

(ii) $p \neq q \Rightarrow$ f is not derivable at $x = a$ but f is continuous at $x = a$

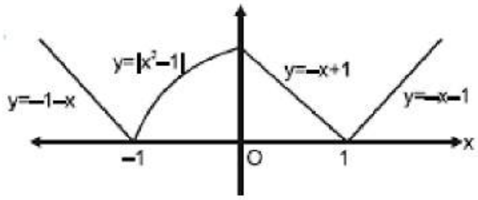
Differentiable $\Rightarrow$ Continuous ; Non-differentiable $\nRightarrow$ Discontinuous

But Discontinuous $\Rightarrow$ Non-differentiable .

(b) If a function f is not differentiable but is continuous at x = a it geometrically implies a sharp corner at x = a.

Ex.15 If $f(x) = \begin{cases} -1-x & ; \; x \leq -1 \\ |x^2-1| & ; \; -1 < x \leq 0 \\ k(-x+1) & ; \; 0 < x \leq 1 \\ |x-1| & ; \; x > 1 \end{cases}$ **, then find the value of k so that f(x) becomes continuous at x = 0. Hence, find all the points where the functions is non-differentiable.**

Sol. From the graph of f(x) it is clear that for the function to be continuous only possible value of k is 1.



Points of non-differentiability are $x = 0, \pm 1$.

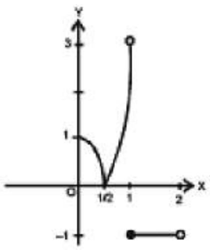
Ex.16 If $f(x) = \begin{cases} |1-4x^2| , & 0 \le x < 1 \\ |x^2 - 2x| , & 1 \le x < 2 \end{cases}$ where [.] denotes the greatest integer function.

Discuss the continuity and differentiability of f(x) in [0, 2).

Sol. Since $1 \le x < 2 \Rightarrow 0 \le x - 1 < 1$ then $[x^2 - 2x] = [(x - 1)^2 - 1] = [(x - 1)^2] - 1 = 0 - 1 = -1$

$$f(x) = \begin{cases} 1-4x^2 , & 0 \le x < \frac{1}{2} \\ 4x^2 - 1 , & \frac{1}{2} \le x < 1 \\ -1 , & 1 \le x < 2 \end{cases}$$

Graph of f(x) :



It is clear from the graph that f(x) is discontinuous at x = 1 and not differentiable at x =1/2,and x= 1

Further details are as follows

$$f(x) = \begin{cases} 1-4x^2 , & 0 \le x < \frac{1}{2} \\ 4x^2 - 1 , & \frac{1}{2} \le x < 1 \\ -1 , & 1 \le x < 2 \end{cases} \Rightarrow f'(x) = \begin{cases} -8x, & 0 \le x < 1/2 \\ 8x, & 1/2 \le x < 1 \\ 0, & 1 \le x < 2 \end{cases}$$

$f'(x) = \begin{cases} -4 & x < 1/2 \\ 4 & x > 1/2 \end{cases}$ and $f'(x) = \begin{cases} 8, & x < 1 \\ 0, & x > 1 \end{cases}$

Hence, which shows f(x) is not differentiable at x = 1/2 (as RHD = 4 and LHD = – 4) and x = 1 (as RHD = 0 and LHD = 8). Therefore, f(x) is differentiable, for x ∈ [0, 2) - {1/2, 1}

Ex.17 Suppose $f(x) = \begin{cases} x^3 & \text{if } x < 1 \\ ax^2 + bx + c & \text{if } x \ge 1 \end{cases}$. If $f''(1)$ exist then find the value of $a^2 + b^2 + c^2$.

Sol. For continuity at x = 1 we leave $f(1^-) = 1$ and $f(1^+) = a + b + c$

$a + b + c = 1$ ....(1)

$f'(x) = \begin{cases} 3x^2 & \text{if } x < 1 \\ 2ax + b & \text{if } x \ge 1 \end{cases}$ for continuity of $f'(x)$ at x = 1 $f'(1^-) = 3$; $f'(1^+) = 2a + b$

hence $2a + b = 3$ ....(2)

$f''(x) = \begin{cases} 6x & \text{if } x < 1 \\ 2a & \text{if } x \ge 1 \end{cases}$ $f''(1-) = 6$; $f''(1+) = 2a$ for continuity of $f''(x)$ $2a = 6 \Rightarrow a = 3$

from (2), b = – 3 ; c = 1. Hence a = 3, b = – 3 ; c = 1

$\therefore \sum a^2 = 19$

Ex.18 Check the differentiability of the function $f(x) = \max\{\sin^{-1}|\sin x|, \cos^{-1}|\sin x|\}$.

Sol. $\sin^{-1}|\sin x|$ is periodic with period $\pi \Rightarrow \sin^{-1}|\sin x|$

$= \begin{cases} x &, \ n\pi \le x \le n\pi + \frac{\pi}{2} \\ \pi - x &, \ n\pi + \frac{\pi}{2} \le x \le n\pi + \pi \end{cases}$

Also $\cos^{-1}|\sin x| = \frac{\pi}{2} - \sin^{-1}|\sin x|$

$$\Rightarrow f(x) = \max \begin{cases} x, \frac{\pi}{2} - x & , \; n\pi \le x \le n\pi + \frac{\pi}{2} \\ \pi - x, x - \frac{\pi}{2} & , \; n\pi + \frac{\pi}{2} \le x \le n\pi + \pi \end{cases}$$

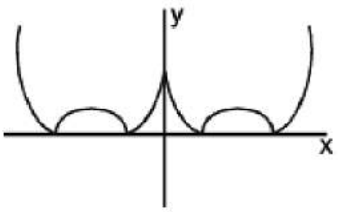
$$\Rightarrow f(x) = \begin{cases} \frac{\pi}{2} - x, & n\pi \le x \le n\pi + \frac{\pi}{4} \\ x, & n\pi + \frac{\pi}{4} < n\pi + \frac{\pi}{2} \\ \pi - x, & n\pi + \frac{\pi}{2} < x \le n\pi + \frac{3\pi}{4} \\ x - \frac{\pi}{2}, & n\pi + \frac{3\pi}{4} < x \le n\pi + \pi \end{cases}$$

$\Rightarrow$ f(x) is not differentiable at $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$.........

$\Rightarrow$ f(x) is not differentiable at $x = \frac{n\pi}{4}$.

Ex.19 Find the interval of values of k for which the function $f(x) = |x^2 + (k - 1)|x| - k|$ is non differentiable at five points.

Sol.



$f(x) = |x^2 + (k - 1)|x| - k| = |(|x| - 1)(|x| + k)|$

Also f(x) is an even function and f(x) is not differentiable at five points.
So $|(x - 1)(x + k)|$ is non differentiable for two positive values of x.

$\Rightarrow$ Both the roots of $(x - 1)(x + k) = 0$ are positive.

$\Rightarrow k < 0 \Rightarrow k \in (-\infty, 0)$.

Definition : A function f is differentiable at a if f'(a) exists. It is differentiable on an open interval (a,b) [or (a, ∝) or (−∝, a) or (− ∝, ∝)] if it is differentiable at every number in the interval.

Derivability Over An Interval : f(x) is said to be derivable over an interval if it is derivable at each & every point of the interval. f(x) is said to be derivable over the closed interval [a, b] if :

(i) for the points a and b, $f'(a^+)$ & $f'(b^-)$ exist &

(ii) for any point c such that $a < c < b$, $f'(c^+)$ & $f'(c^-)$ exist & are equal .

Limit and Continuity & Differentiability of Function Formulas

Things To Remember :

1. Limit of a function f(x) is said to exist as, x→a when $\underset{x\to a^-}{\text{Limit}}\ f(x) = \underset{x\to a^+}{\text{Limit}}\ f(x) =$ finite quantity.

2. Fundamental Theorems On Limits:

Let $\underset{x\to a}{\text{Limit}}\ f(x) = l$ & $\underset{x\to a}{\text{Limit}}\ g(x) = m$. If l & m exists then :

(i) $\underset{x\to a}{\text{Limit}}\ f(x) \pm g(x) = l \pm m$

(ii) $\underset{x\to a}{\text{Limit}}\ f(x) \cdot g(x) = l.\, m$

(iii) $\underset{x\to a}{\text{Limit}}\ \frac{f(x)}{g(g)} = \frac{\ell}{m}$, provided $m \neq 0$

(iv) $\underset{x\to a}{\text{Limit}}\ kf(x) = k\ \underset{x\to a}{\text{Limit}}\ f(x)$; where k is a constant.

(v) $\underset{x\to a}{\text{Limit}}\ f[g(x)] = f\left(\underset{x\to a}{\text{Limit}}\ g(x)\right)$ = f (m) ; provided f is continuous at g (x) = m.

For example $\underset{x\to a}{\text{Limit}}\ l\text{n}\,(f(x) = l\text{n}\left[\underset{x\to a}{\text{Limit}}\ f(x)\right]\ l\text{n}\ l\ (l > 0).$

3. Standard Limits :

(a) $\underset{x\to 0}{\text{Limit}}\ \frac{\sin x}{x} = 1 = \underset{x\to 0}{\text{Limit}}\ \frac{\tan x}{x} = \underset{x\to 0}{\text{Limit}}\ \frac{\tan^{-1}x}{x} = \underset{x\to 0}{\text{Limit}}\ \frac{\sin^{-1}x}{x}$ [Where x is measured in radians]

(b) $\underset{x\to 0}{\text{Limit}}\ (1+x)^{1/x} = e = \underset{x\to\infty}{\text{Limit}}\ \left(1+\frac{1}{x}\right)^x$ note however the re $\underset{\substack{h\to 0\\ n\to\infty}}{\text{Limit}}\ (1 - h)^n$ = 0and

$$\underset{\substack{h\to 0\\ n\to\infty}}{\text{Limit}}\ (1+h)^n \to \infty$$

(c) If $\underset{x\to a}{\text{Limit}}\ f(x) = 1$ and $\underset{x\to a}{\text{Limit}}\ \Phi(x) = \infty$, then ;

$$\underset{x\to a}{\text{Limit}}\ [f(x)]^{\phi(x)} = e^{\underset{x\to a}{\text{Limit}}\ \phi(x)[f(x)-1]}$$

(d) If $\underset{x\to a}{\text{Limit}}\ f(x) = A > 0$ & $\underset{x\to a}{\text{Limit}}\ \Phi(x) = B$ (a finite quantity) then ;

$\underset{x\to a}{\text{Limit}}\ [f(x)]^{\phi(x)} = e^z$ where $z = \underset{x\to a}{\text{Limit}}\ \Phi(x).\ln[f(x)] = e^{B\ln A} = A^B$

(e) $\underset{x\to 0}{\text{Limit}}\ \frac{a^x - 1}{x} = 1\text{n a}\ (a > 0)$. In particular $\underset{x\to 0}{\text{Limit}}\ \frac{e^x - 1}{x} = 1$

(f) $\underset{x\to a}{\text{Limit}}\ \frac{x^n - a^n}{x - a} = na^{n-1}$

4. Squeeze Play Theorem :

If $f(x) \le g(x) \le h(x)\ \forall\ x$ & $\underset{x \to a}{\text{Limit}} f(x) = l = \underset{x \to a}{\text{Limit}} h(x)$ then $\underset{x \to a}{\text{Limit}} g(x) = l$.

5. Indeterminant Forms :

$$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, 0^{\circ}, \infty^{\circ}, \infty - \infty \text{ and } 1^{\infty}$$

Note :

(i) We cannot plot ∞ on the paper. Infinity (∞) is a symbol & not a number. It does not obey the laws of elementry algebra.

(ii) $\infty + \infty = \infty$

(iii) $\infty \times \infty = \infty$

(iv) $(a/\infty) = 0$ if a is finite

(v) $a/0$ is not defined, if $a \neq 0$.

(vi) $ab = 0$, if & only if $a = 0$ or $b = 0$ and a & b are finite.

6. The following strategies should be born in mind for evaluating the limits:

(a) Factorisation

(b) Rationalisation or double rationalisation

(c) Use of trigonometric transformation ; appropriate substitution and using standard limits

(d) Expansion of function like Binomial expansion, exponential & logarithmic expansion, expansion of sinx , cosx , tanx should be remembered by heart & are given below :

(i) $a^x = 1 + \frac{x \ln a}{1!} + \frac{x^2 \ln^2 a}{2!} + \frac{x^3 \ln^3 a}{3!} + \ldots\ldots\ldots a > 0$

(ii) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots$

(iii) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots \text{for} -1 < x \leq$

(iv) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots.$

(v) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots$

(vi) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots\ldots$

(vii) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots.$

(viii) $\sin^{-1} x = x + \frac{1^2}{3!} x^3 + \frac{1^2.3^2}{5!} x^5 + \frac{1^2.3^2.5^2}{7!} x^7 + \ldots\ldots.$

(ix) $\sec^{-1} x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \ldots\ldots$

(Continuity)

Things To Remember :

1. A function f(x) is said to be continuous at x = c, if $\underset{x \to c}{\text{Limit}} f(x) = f(c)$. Symbolically
f is continuous at x = c if $\underset{h \to 0}{\text{Limit}} f(c - h) = \underset{h \to 0}{\text{Limit}} f(c+h) = f(c)$.
i.e. LHL at x = c = RHL at x = c equals Value of 'f' at x = c.
It should be noted that continuity of a function at x = a is meaningful only if the function is defined in the immediate neighbourhood of x = a, not necessarily at x = a.

2. Reasons of discontinuity:

(i) $\underset{x \to c}{\text{Limit}} f(x)$ does not exist
i.e. $\underset{x \to c^-}{\text{Limit}} f(x) \neq \underset{x \to c^+}{\text{Limit}} f(x)$
(ii) f(x) is not defined at x= c
(iii) $\underset{x \to c}{\text{Limit}} f(x) \neq f(c)$

Geometrically, the graph of the function will exhibit a break at x= c.

The graph as shown is discontinuous at x = 1 , 2 and 3.

3. Types of Discontinuities :

Type - 1: (Removable type of discontinuities)

In case $\lim_{x \to c} f(x)$ exists but is not equal to f(c) then the function is said to have a removable discontinuity or discontinuity of the first kind. In this case we can redefine the function such that $\lim_{x \to c} f(x) = f(c)$ & make it continuous at x= c.
Removable type of discontinuity can be further classified as :

(a) Missing Point Discontinuity : Where $\lim_{x \to a} f(x)$ exists finitely but f(a) is not defined.

e.g. $f(x) = \frac{(1-x)(9-x^2)}{(1-x)}$ has a missing point discontinuity at x = 1 , and $f(x) = \frac{\sin x}{x}$ has a missing point discontinuity at x = 0

(b) Isolated Point Discontinuity : Where $\lim_{x \to a} f(x)$ exists & f(a) also exists but $\lim_{x \to a} \neq f(a)$. e.g. $f(x) = \frac{x^2 - 16}{x - 4}$, $x \neq 4$ & $f(4) = 9$ has an isolated point discontinuity at x = 4.

Similarly $f(x) = [x] + [-x] = \begin{cases} 0 & \text{if } x \in I \\ -1 & \text{if } x \notin I \end{cases}$ has an isolated point discontinuity at all x ∈ I.

Type-2: (Non - Removable type of discontinuities)

In case $\lim_{x \to c} f(x)$ does not exist then it is not possible to make the function continuous by redefining it.

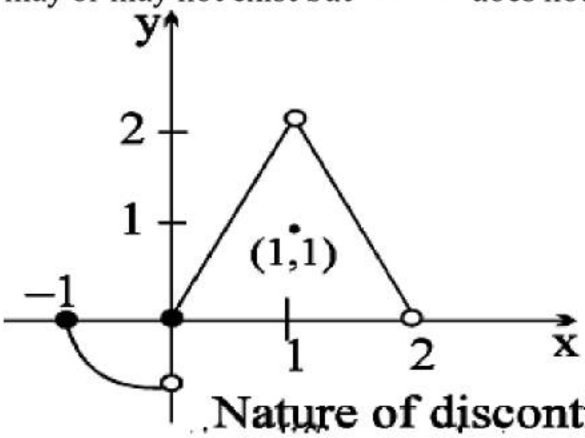
Such discontinuities are known as non - removable discontinuity or discontinuity of the 2nd kind. Non-removable type of discontinuity can be further classified as :

(a) Finite discontinuity e.g. $f(x) = x - [x]$ at all integral x ; $f(x) = \tan^{-1}\frac{1}{x}$ at $x = 0$ and $f(x) = \frac{1}{1+2^{\frac{1}{x}}}$ at $x = 0$ (note that $f(0^+) = 0$; $f(0^-) = 1$)

(b) Infinite discontinuity e.g. $f(x) = \frac{1}{x-4}$ or $g(x) = \frac{1}{(x-4)^2}$ at $x = 4$; $f(x) = 2^{\tan x}$ at $x = \frac{\pi}{2}$ and $f(x) = \frac{\cos x}{x}$ at $x = 0$.

(c) Oscillatory discontinuity e.g. $f(x) = \sin\frac{1}{x}$ at $x = 0$.

In all these cases the value of f(a) of the function at x= a (point of discontinuity) may or may not exist but $\lim_{x \to a}$ does not exist.



Nature of discontinuity

Note: From the adjacent graph note that
- f is continuous at $x = -1$
- f has isolated discontinuity at $x = 1$
- f has missing point discontinuity at $x = 2$
- f has non removable (finite type) discontinuity at the origin.

4. In case of dis-continuity of the second kind the non-negative difference between the value of the RHL at x = c & LHL at x = c is called **The Jump Of Discontinuity.** A function having a finite number of jumps in a given interval I is called a **Piece Wise Continuous Or Sectionally Continuous** function in this interval.

5. All Polynomials, Trigonometrical functions, exponential & Logarithmic functions are continuous in their domains.

6. If f & g are two functions that are continuous at x= c then the functions defined by :

$F_1(x) = f(x) \pm g(x)$; $F_2(x) = K\, f(x)$, K any real number ; $F_3(x) = f(x).g(x)$ are also continuous at x= c.

Further, if g (c) is not zero, then $F_4(x) = \frac{f(x)}{g(x)}$ is also continuous at x= c.

7. The intermediate value theorem:

Suppose f(x) is continuous on an interval I , and a and b are any two points of I. Then if y_0 is a number between f(a) and f(b) , their exists a number c between a and b such that $f(c) = y_0$.

Note Very Carefully That :

(a) If f(x) is continuous & g(x) is discontinuous at x = a then the product function $\varphi(x) = f(x).\, g(x)$ is not necessarily be discontinuous at x = a. e.g. $f(x) = x$ & $g(x) = \begin{cases} \sin\frac{\pi}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

(b) If f(x) and g(x) both are discontinuous at x = a then the product function $\varphi(x) = f(x).\, g(x)$ is not necessarily be discontinuous at x = a. e.g. $f(x) = -g(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$

(c) Point functions are to be treated as discontinuous. eg. $f(x) = \sqrt{1-x} + \sqrt{x-1}$ is not continuous at x = 1.

(d) A Continuous function whose domain is closed must have a range also in closed interval.

(e) If f is continuous at x = c & g is continuous at x = f(c) then the composite g[f(x)] is continuous at x = c. eg. $f(x) = \frac{x \sin x}{x^2+2}$ & $g(x) = |x|$ are continuous at x = 0

, hence the composite (gof) (x) = $\left|\frac{x \sin x}{x^2+2}\right|$ will also be continuous at x = 0 .

7. Continuity In An Interval :

(a) A function f is said to be continuous in (a , b) if f is continuous at each & every point ∈(a,b).
(b) A function f is said to be continuous in a closed interval [a , b] if :
(i) f is continuous in the open interval (a , b) &
(ii) f is right continuous at 'a' i.e. $\underset{x \to a^+}{\text{Limit}}\ f(x) = f(a)$ = a finite quantity.
(iii) f is left continuous at 'b' i.e. $\underset{x \to b^-}{\text{Limit}}\ f(x) = f(b)$ = a finite quantity.
Note that a function f which is continuous in [a , b] possesses the following properties :

(i) If f(a) & f(b) possess opposite signs, then there exists at least one solution of the equation f(x) = 0 in the open interval (a , b).

(ii) If K is any real number between f(a) & f(b), then there exists at least one solution of the equation f(x) = K in the open inetrval (a , b).

8. Single Point Continuity:
Functions which are continuous only at one point are said to exhibit single point continuity

e.g. $f(x) = \begin{cases} x & \text{if } x \in Q \\ -x & \text{if } x \notin Q \end{cases}$ and $g(x) = \begin{cases} x & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{cases}$ are both continuous only at x = 0.

Differentiability

Things To Remember :

1. Right hand & Left hand Derivatives ; By definition:

$$f'(a) = \underset{h \to 0}{\text{Limit}} \frac{f(a+h)-f(a)}{h}$$ if it exist

(i) The right hand derivative of f ′ at x = a denoted by $f'(a^+)$ is defined by :

$$f'(a^+) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h},$$

provided the limit exists & is finite.

(ii) The left hand derivative : of f at x = a denoted by

$f'(a^+)$ is defined by : $f'(a^-) = \lim_{h\to 0^+} \frac{f(a-h)-f(a)}{-h}$, Provided the limit exists & is finite.

We also write $f'(a+) = f'_+(a)$ & $f'(a-) = f'_-(a)$.

* This geomtrically means that a unique tangent with finite slope can be drawn at x = a as shown in the figure.

(iii) Derivability & Continuity :

(a) If f ′(a) exists then f(x) is derivable at x= a ⇒ f(x) is continuous at x = a.

(b) If a function f is derivable at x then f is continuous at x.

For : $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists.

Also $f(x+h)-f(x) = \frac{f(x+h)-f(x)}{h}\, h\,[h \neq 0]$

Therefore : $[f(x+h)-f(x)] = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \cdot h = f'(x).0 = 0$

Therefore $\lim_{h\to 0} [f(x+h)-f(x)] = 0 \Rightarrow \lim_{h\to 0} f(x+h) = f(x) \Rightarrow$ f is continuous at x.

Note : If f(x) is derivable for every point of its domain of definition, then it is continuous in that domain.

The Converse of the above result is not true:

" IF f IS CONTINUOUS AT x , THEN f IS DERIVABLE AT x " IS NOT TRUE.

e.g. the functions $f(x) = |x|$ & $g(x) = x \sin \frac{1}{x}$; $x \neq 0$ & $g(0) = 0$ are continuous at x = 0 but not derivable at x = 0.

Note Carefully :

(a) Let $f'_{+}(a) = p$ & $f'_{-}(a) = q$ where p & q are finite then :
(i) $p = q \Rightarrow$ f is derivable at $x = a \Rightarrow$ f is continuous at $x = a$.
(ii) $p \neq q \Rightarrow$ f is not derivable at $x = a$.

It is very important to note that f may be still continuous at $x = a$.

In short, for a function f :

Differentiability $\Rightarrow$ Continuity ; Continuity $\not\Rightarrow$ derivability ;
Non derivibality $\not\Rightarrow$ discontinuous ; But discontinuity $\Rightarrow$ Non derivability

(b) If a function f is not differentiable but is continuous at $x = a$ it geometrically implies a sharp corner at $x = a$.

3. Derivability Over An Interval :
f (x) is said to be derivable over an interval if it is derivable at each & every point of the interval f(x) is said to be derivable over the closed interval [a, b] if :

(i) for the points a and b, $f'(a+)$ & $f'(b-)$ exist &
(ii) for any point c such that $a < c < b$, $f'(c+)$ & $f'(c-)$ exist & are equal.
Note:

1. If f(x) & g(x) are derivable at $x = a$ then the functions f(x) + g(x), f(x) - g(x) , f(x).g(x) will also be derivable at $x = a$ & if $g(a) \neq 0$ then the function f(x)/g(x) will also be derivable at $x = a$.

2. If f(x) is differentiable at $x = a$ & g(x) is not differentiable at $x = a$, then the product function $F(x) = f(x)\, g(x)$ can still be differentiable at $x = a$ e.g. $f(x) = x$ & $g(x) = |x|$.

3. If f(x) & g(x) both are not differentiable at $x = a$ then the product function ;

$F(x) = f(x)\cdot g(x)$ can still be differentiable at $x = a$ e.g. $f(x) = |x|$ & $g(x) = |x|$.

4. If f(x) & g(x) both are non-deri. at $x = a$ then the sum function $F(x) = f(x) + g(x)$ may be a differentiable function. e.g. $f(x) = |x|$ & $g(x) = -|x|$.

5. If f(x) is derivable at $x = a \not\Rightarrow f'(x)$ is continuous at $x = a$.

$$\text{e.g. } f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

6. A surprising result : Suppose that the function f (x) and g (x) defined in the interval (x_1, x_2) containing the point x_0, and if f is differentiable at $x = x_0$ with $f(x_0)$

$= 0$ together with g is continuous as $x = x_0$ then the function $F(x) = f(x) \cdot g(x)$ is differentiable at $x = x_0$ e.g. $F(x) = \sin x \cdot x^{2/3}$ is differentiable at $x = 0$.